Homological algebra solutions Week 7

1.

- We will prove $(a) \implies (b) \implies (c) \implies (a)$.

$(a) \implies (b)$: Let A be a left R-module and $P_\bullet \to A$ a projective resolution of A. By definition the complex

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

is exact, so by flatness of B the tensored complex

$$(P_\bullet \otimes B) = \cdots \to P_2 \otimes B \to P_1 \otimes B \to P_0 \otimes B$$

is exact as well. It follows that the homology groups of this complex are zero and so $\mathrm{Tor}_n^R(A,B) = H_n(P_\bullet \otimes B) = 0$ for every $n > 0$.

$(b) \implies (c)$: immediate.

$(c) \implies (a)$: Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules and consider its associated long exact sequence ($\mathrm{Tor}_\bullet(-,B)$ is a homological δ-functor)

$$\cdots \to \mathrm{Tor}_1(M_3,B) \to M_1 \otimes B \to M_2 \otimes B \to M_3 \otimes B \to 0$$

where we used that $\mathrm{Tor}_0(A,B) = A \otimes B$ for all A. Since $\mathrm{Tor}_1(M_3,B) = 0$ by hypothesis, the functor $- \otimes B$ is exact, as desired.

- Using that $\mathrm{Tor}_\bullet(M,-) \cong L_\bullet(M \otimes -)(-)$ and the long exact sequence for this homological δ-functor associated to the mentioned short exact sequence, we obtain directly from the first part of the exercise that $0 \to \mathrm{Tor}_1(M,B) \to 0$ is an exact sequence for all R-module M, i.e. $\mathrm{Tor}_1(M,B) = 0$ for all M. We conclude by the first part.

2. Let $0 \to M \to P \to A \to 0$ be a short exact sequence where P is $\mathcal{F}$-acyclic.

(a) Using the long exact sequence for $L_\bullet\mathcal{F}$ associated to the above short exact sequence, using that $L_n\mathcal{F}(P) = 0$ for all $n > 0$ and that $L_0\mathcal{F}(B) = \mathcal{F}(B)$, yields directly the statement.

(b) Consider the exact sequence

$$0 \to M_m \xrightarrow{f_{m+1}} P_m \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \xrightarrow{f_{-1}} 0 \quad (1)$$

and split it into short exact sequences

$$0 \to K_j \to P_j \xrightarrow{f_j} K_{j-1} \to 0 \tag{2}$$

for all $0 \leq j \leq m$, where $K_j = \ker f_j = \mathrm{im} f_{j+1}$. Note that $K_{-1} = A$ and $K_m = M_m$. Using this and the previous point, we obtain by induction that

$$\begin{aligned} L_i\mathcal{F}(A) &\cong L_{i-0-1}\mathcal{F}(K_0) \cong L_{i-1-1}\mathcal{F}(K_1) \cong \dots \\ &\cong L_{i-m-1}\mathcal{F}(K_m) = L_{i-m-1}\mathcal{F}(M_m) \end{aligned}$$

for $i \geq m+2$, as desired. When $i = m+1$ we use the same sequence of isomorphisms, but use the previous point in the last step to get:

$$L_{m+1}\mathcal{F}(A) \cong \cdots \cong L_1\mathcal{F}(K_{m-1}) = \ker(\mathcal{F}(M_m) \to \mathcal{F}(P_m)).$$

(c) Let $P_\bullet \to A$ be an $\mathcal{F}$-acyclic resolution of A. Split this exact sequence in short exact sequences of the form (2) for all $j \geq 0$ (with the same notations). For $i \geq 1$, let $m = i-2$ and set $M_m = M_{i-2} = K_{i-2} = \ker f_{i-2}$ and consider the exact sequence (1). Applying the previous point yields that

$$L_i\mathcal{F}(A) \cong L_1\mathcal{F}(K_{i-2}) = \ker(\mathcal{F}K_{i-1} \to \mathcal{F}P_{i-1}) \tag{3}$$

Since $\mathcal{F}$ is right exact it preserves cokernels, hence we obtain that

$$\begin{aligned} \mathcal{F}K_{i-1} &= \mathcal{F}(\mathrm{im} f_i) \cong \mathcal{F}(P_i/\ker f_i) = \mathcal{F}(P_i/\mathrm{im} f_{i+1}) \\ &= \mathcal{F}(\mathrm{coker} f_{i+1}) \cong \mathrm{coker}(\mathcal{F}f_{i+1}) \end{aligned}$$

Continuing the sequence (3) of isomorphisms, it follows that

$$L_i\mathcal{F}(A) \cong \ker\left(\mathcal{F}P_i/\mathrm{im}(\mathcal{F}f_{i+1}) \to \mathcal{F}P_{i-1}\right) \cong H_i(\mathcal{F}P_\bullet)$$

as desired.

3. We use induction to prove that $\mathrm{Tor}_n(A, B) \cong H_n(F \otimes B)$ for all n. For $n = 0$, we have that $F_1 \otimes B \to F_0 \otimes B \to A \otimes B \to 0$ is exact since $\otimes B$ is right exact. Therefore we have that

$$\mathrm{Tor}_0(A, B) = A \otimes B \cong (F_0 \otimes B)/Im(F_1 \otimes B) = H_0(F \otimes B)$$

For $n = 1$, let $K_0 := Ker(F_0 \to A)$ and consider the following exact sequences

$$0 \to K_0 \to F_0 \to A \to 0 \tag{4}$$

$$F_2 \to F_1 \to K_0 \to 0 \tag{5}$$

Using the long exact sequence for $L_\bullet(- \otimes B)$ on the short exact sequence (4) and using that $\mathrm{Tor}_1(F_0, B) = 0$ by the first exercise, we have that

$$\mathrm{Tor}_1(A, B) \cong Ker(K_0 \otimes B \to F_0 \otimes B).$$

Using that the image of (5) under the functor $- \otimes B$ is still exact, the right hand side of the above equation is

$$Ker(F_1 \otimes B/Im(F_2 \otimes B) \to F_0 \otimes B) = H_1(F_\bullet \otimes B)$$

as desired. For $n \geq 2$ notice that $F_{\bullet+1} \to K_0$ is a flat resolution of K_0. Hence by induction and using the dimension shifting exercise on (4) (this makes sense since F_0 is flat, thus $(- \otimes B)$-acyclic by the first exercise) we find that

$$\begin{aligned}\mathrm{Tor}_n(A, B) &= L_n(- \otimes_R B)(A) \cong L_{n-1}(- \otimes_R B)(K_0) \\ &= \mathrm{Tor}_{n-1}(K_0, B) = H_{n-1}(F_{\bullet+1} \otimes B) \\ &= H_n(F_\bullet \otimes B)\end{aligned}$$

4. (a) Since $C_{p,q} = 0$ whenever $q < 0$ we have isomorphisms at each level

$$\mathrm{Tot}^{\prod}(C)_n = \prod_{q \geq 0} C_{n-q,q} \cong \prod_{q \geq 0} \mathbb{Z}/4\mathbb{Z}$$

Through these isomorphisms, the differentials at each level are all the same and are given by

$$\left(a_q\right)_{q \geq 0} \mapsto \left(2a_q + 2a_{q+1}\right)_{q \geq 0}.$$

Let us show that $im(d) = \prod_{q \geq 0} 2\mathbb{Z}/4\mathbb{Z}$. Clearly the left hand side is included in the right hand side. Now let $(a_q)_{q \geq 0} \in \prod_{q \geq 0} 2\mathbb{Z}/4\mathbb{Z}$. We can build $(b_q)_{q \geq 0}$ inductively with ones and zeros by setting $b_0 = \frac{a_0}{2}$ and

$$b_q = \begin{cases} 0 & \text{if } a_{q-1} = b_{q-1} = 0 \text{ or } (a_{q-1} = 2 \wedge b_{q-1} = 1); \\ 1 & \text{if } (a_{q-1} = 0 \wedge b_{q-1} = 1) \text{ or } (a_{q-1} = 2 \wedge b_{q-1} = 0) \end{cases}$$

This element is mapped to $(a_q)_{q \geq 0}$ by construction which shows the hint.

Moreover we can easily observe that $(a_q)_{q \geq 0}$ is a cycle if and only if it is a boundary ($a_q \in \{0, 2\}$ for all q) or if it is of the form $a_q \in \{1, 3\}$ for all q. Since any two cycles that both fall in the second case differ by a boundary (an element of $\prod 2\mathbb{Z}/4\mathbb{Z}$), we obtain that the quotient $\ker(d)/im(d) = H_0(Tot^{\prod}(C))$ has two classes (generated by a boundary and by $(1, 1, \ldots)$ for example), i.e. it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(b) Since the rows of C are exact, the acyclic assembly lemma shows that that $Tot^{\oplus}(C)$ is acyclic. Another way of seeing this directly is by observing that the cycles in $Tot^{\prod}(C)$ are either boundaries, either sequences of elements in $\{1,3\}$. Since the latter are not elements of the direct product total complex $Tor^{\oplus}(C)$, the boundaries are equal to the cycles.

(c) The same formulas hold in this case, except that the products are indexed by $q \in \mathbb{Z}$ instead of positive integers. Again the element $x = (\dots, 1, 1, 1, \dots)$ is a cycle, but can't be a boundary since those are precisely $\prod_{\mathbb{Z}} 2\mathbb{Z}/4\mathbb{Z}$. Hence the total complex of D is not acyclic. Moreover there is a chain map $Tot^{\prod}(D) \to Tot^{\prod}(C)$ given at each level by $(a_q)_{q\in\mathbb{Z}} \mapsto (a_q)_{q\geq 0}$. It induces a surjective map on the 0-th homology groups

$$H_0(Tot^{\prod}(D)) \to H_0(Tot^{\prod}(C)) \cong \mathbb{Z}/2\mathbb{Z}.$$

Indeed, the class of the cycle $(1)_{q\in\mathbb{Z}}$ is mapped to the class of the cycle $(1)_{q\geq 0}$, which corresponds to the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$. Lastly, the element $(a_q)_{q\in\mathbb{Z}}$ defined by $a_q = 0$ for all $q \neq 0$ and $a_0 = 2$ is a cycle, but can't be a boundary. The only preimage of such an element in $Tot^{\prod}(D)$ are of the form $(\dots, b_3, b_2, b_1, 0, 0, 0, \dots)$ or $(\dots, 0, 0, 0, b_0, b_{-1}.b_{-2}, \dots)$ for $b_i \in \{1,3\}$. Those possible elements are not in $Tot^{\oplus}(D)$, so (a_q) is not a boundary and the homology groups are non trivial.